University of BRISTOL

Wave Propagation Theory

Derivation and general form of wave equation

Outline

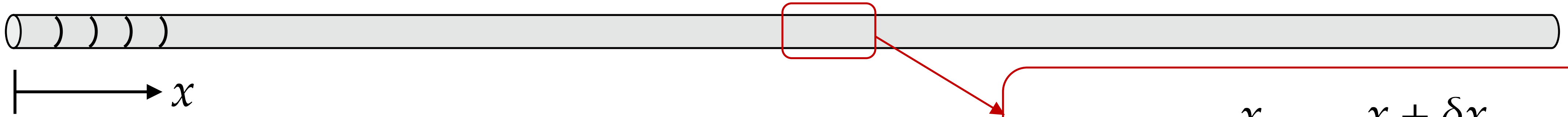
- Elastodynamic wave equation
- Solutions of wave equation
- Modelling wave propagation
  - Linear systems theory
  - Numerical modelling

Wave theory

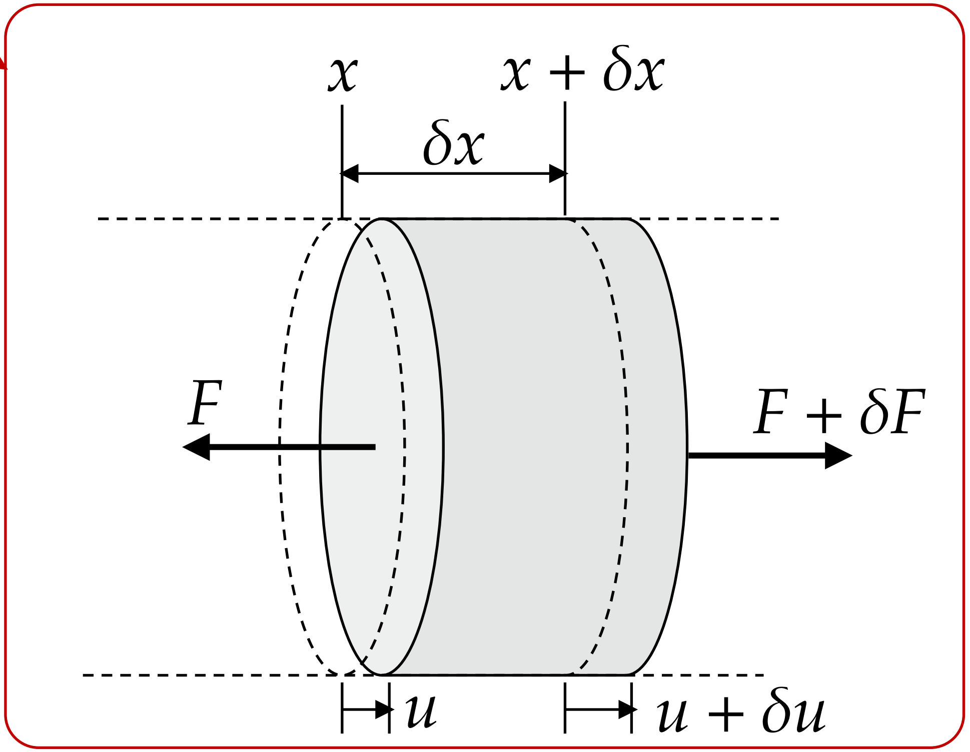
- All wave propagation is governed by wave equations
- Solved by appropriate boundary conditions

Elastodynamic Wave Equation in 1D

- Compression waves in an elastic bar



  - Distance along bar = x, time = t
  - Cross sectional area = A, density = ρ
  - Particle displacement = $u\,(x,t)$
  - Axial force (tension is positive) = $F(x,t)$



- Consider a small length of bar at an instant in time:

Elastodynamic Wave Equation in 1D

- Force equilibrium
  - Net force on element to right

$$(F + \delta F) - F = \delta F$$

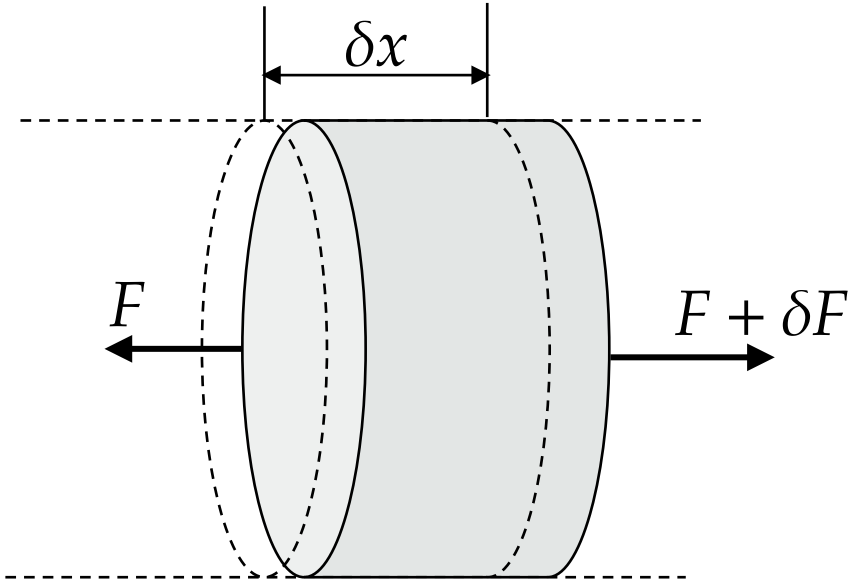
  - Mass of element $= \rho A \delta x$ (mass conserved)
  - Acceleration of element $= \dfrac{\partial^2 u}{\partial t^2}$ (higher order terms ignored)
  - Newton's second law

$$\delta F = \rho A \delta x \frac{\partial^2 u}{\partial t^2}$$

  - and also

$$\delta F = \frac{\partial F}{\partial x} \delta x$$

(assumes δx is small)



Elastodynamic Wave Equation in 1D

- Elasticity
  - Hooke's Law for bar relates stress, σ, to strain, ε, and Young's modulus, E

$$\sigma = E\varepsilon$$

(linear stress-strain relation assumed)

  - Strain displacement relationship

$$\varepsilon = \frac{\partial u}{\partial x}$$

(small strains assumed)

  - Hence force and displacement related by

$$F = A\sigma = AE\varepsilon = AE\frac{\partial u}{\partial x}$$

Elastodynamic Wave Equation in 1D

- From previous slides we have

$$\delta F = \rho A \delta x \frac{\partial^2 u}{\partial t^2}, \;\; \delta F = \frac{\partial F}{\partial x}\delta x, \;\; F = AE\frac{\partial u}{\partial x}$$

(equilibrium) (elasticity, stress-strain)

- Combine these

$$\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right)\delta x = \rho A \delta x \frac{\partial^2 u}{\partial t^2}$$

- and simplify

$$E\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}$$

Elastodynamic Wave Equation in 3D

- Same principle can be used to derive 3 coupled PDEs for displacement components, u, v and w, in x, y and z in isotropic elastic solid

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu) \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho \frac{\partial^2 v}{\partial t^2} = (\lambda + \mu) \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho \frac{\partial^2 w}{\partial t^2} = (\lambda + \mu) \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

where λ and μ are Lamé constants are alternative way of writing elastic properties

$$\lambda = \frac{Ev}{(1+v)(1-2v)}, \quad \mu = \frac{E}{2(1+v)}$$

where E is Young's modulus and v is Poisson's ratio

Elastodynamic Wave Equation in 3D

– Tensor notation for anisotropic elastic solids

$$\rho \frac{\partial^2 u_j}{\partial t^2} = \frac{1}{2} c_{ijkl} \frac{\partial}{\partial x_i} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)$$

where $cijkl$ is stiffness tensor (tensor representation of anisotropic equivalent of Young's modulus and Poisson's ratio)

Note: again there will be 3 PDEs (for $j = 1,2,3$) and Einstein convention implies summations over all possible combinations of i, k and l (i.e. 3x3x3=27 terms in each PDE if written out in full!)

- General form of wave equation is always

  [density] x [second derivative w.r.t. time] = [stiffness] x [second derivative w.r.t. space]

Solution of Wave Equation

- Wave equation is second order PDE which in 1D looks like this

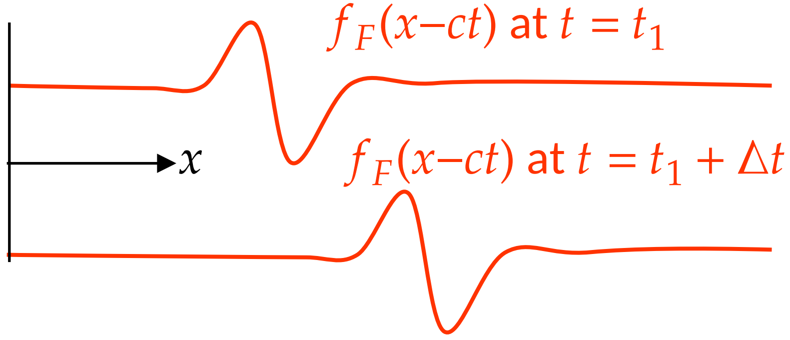
$$\rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2}$$

- To solve it for a particular situation means finding a displacement field that satisfies:
  1. The wave equation (i.e. the above PDE)
  2. The appropriate boundary conditions
- Analytical solutions that satisfy 1 and 2 only possible in limited cases
- Modal solution of 1 enables possible wave modes to be identified

Solution of 1D Wave Equation

- First consider 1D wave equation for waves in a bar:

$$\rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2}$$



- The general solution to this can be written (proof on next slide)

$$u(x,t) = f_F(x{-}ct) + f_B(-x{-}ct)$$

where f_F and f_B are <u>arbitrary</u> functions and represent wave shapes that propagate without distortion in the forwards and backwards directions respectively, and c turns out to be wave velocity

Solution of 1D Wave Equation

– Proof that solution works:

$$\frac{\partial^2 u}{\partial x^2} = f_F{}''(x-ct) + f_B{}''(-x-ct)$$

Use of chain rule

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[f_F{}''(x-ct) + f_B{}''(-x-ct)\right]$$

Use of chain rule

which obviously satisfies something in same form as wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

– Compare with wave equation to find

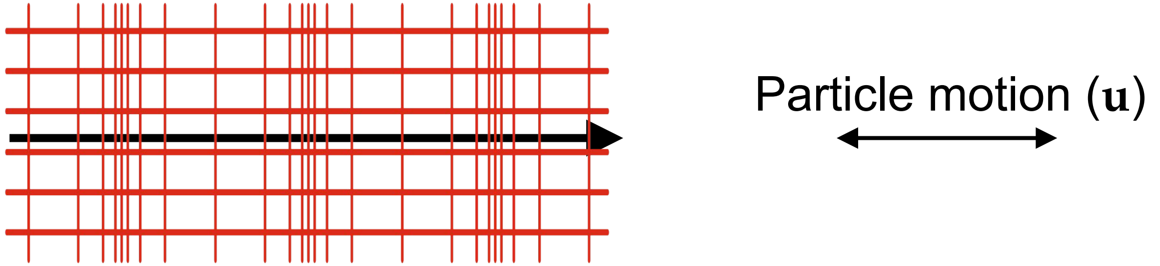
$$c = \sqrt{\frac{E}{\rho}}$$

Wave velocity always has this general relationship to material properties:

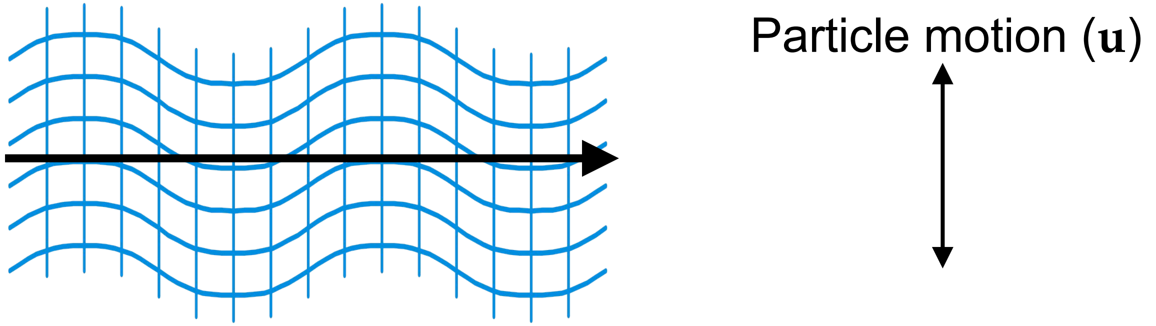
square root of stiffness over density

Solution of 3D Wave Equation

- Modal solution
- The two wave modes
  - Longitudinal (or compression or pressure), velocity c_L



  - Transverse (or shear), velocity c_T (or c_S)



Solution of 3D Wave Equation

- Some typical wave velocities

Wave mode	Equation	Steel $(E = 210\ \text{MPa},\ \nu = 0.3,$ $\rho = 7{,}800\ \text{kg m}^{-3})$	Aluminium $(E = 70\ \text{MPa},\ \nu = 0.33,$ $\rho = 2{,}700\ \text{kg m}^{-3})$
Bar	$\sqrt{\frac{E}{\rho}}$	$5{,}189\ \text{ms}^{-1}$	$5{,}092\ \text{ms}^{-1}$
Longitudinal bulk	$\sqrt{\frac{E(1-v)}{\rho(1+v)(1-2v)}}$	$6{,}020\ \text{ms}^{-1}$	$6{,}236\ \text{ms}^{-1}$
Shear bulk	$\sqrt{\frac{E}{2\rho(1+v)}}$	$3{,}218\ \text{ms}^{-1}$	$3{,}118\ \text{ms}^{-1}$

Satisfying Simple Boundary Condition

- Superposition
  - The wave equation is linear hence if f_1 and f_2 are two solutions then $f_1 + f_2$ is also a solution
  - Real problems are solved by combining simple solutions to satisfy the necessary boundary conditions
- Boundary conditions
  - Wave equation governs what goes on within the material
  - Solutions must satisfy this <u>and</u> what goes on at the edges
  - Examples
    - No displacement if soft material has a rigid boundary ($u = 0$)
    - No stress on a free surface (e.g. $du/dx = 0$)

Satisfying Simple Boundary Condition

- Boundary condition example – waves in bar with free end at $x = 0$
  - Solution for wave in bar has this form:

$$u(x,t) = Ae^{i(kx-\omega t)} + Be^{i(-kx-\omega t)}$$

  - Stress $\sigma = E\varepsilon$ anywhere in bar is:

$$\sigma(x,t) = ikEAe^{i(kx-\omega t)} - ikEBe^{i(-kx-\omega t)}$$

  - The stress at free end ($x = 0$) is zero

$$\sigma(x=0,t) = ikE(A-B)e^{-i\omega t} = 0$$

  - Therefore $A = B$

Satisfying Simple Boundary Condition

- Hence final solution (in terms of displacement) is:

$$u(x,t) = A\left[e^{i(kx-\omega t)} + e^{i(-kx-\omega t)}\right] = A\left[e^{ikx} + e^{-ikx}\right]e^{-i\omega t}$$

  - Physically it is two waves going in opposite directions with the same amplitude – one is the reflection from the end

- Above expression can be expanded to show that the result is a standing wave:

$$u(x,t) = 2A\cos(kx)\,e^{-i\omega t}$$

- And the stress is

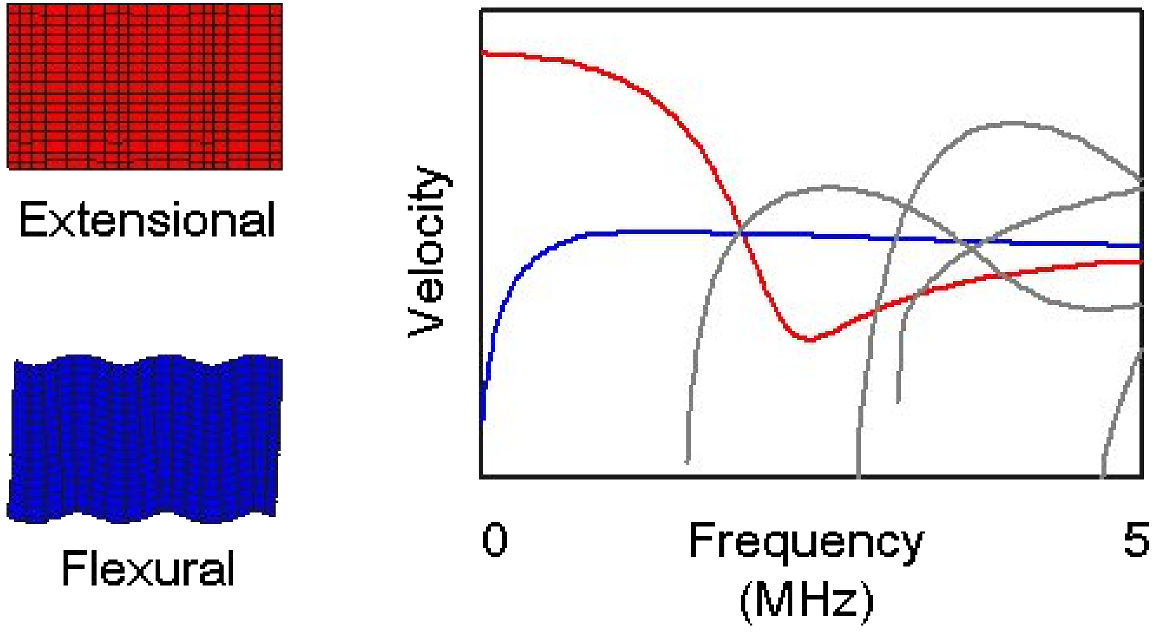
$$\sigma(x,t) = -2AEk\sin(kx)\,e^{-i\omega t}$$

  - Note stress is maximum where displacement is minimum and *vice versa*

More complex boundary conditions

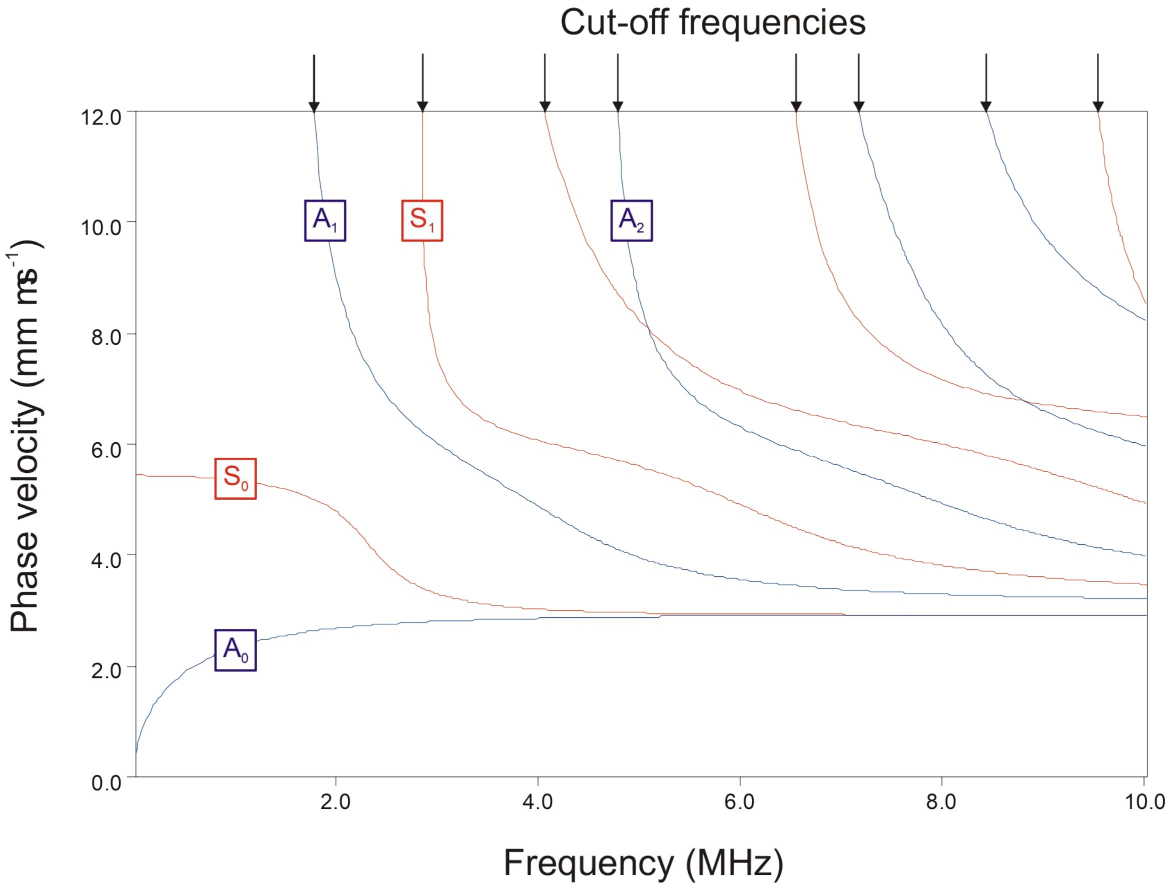
- Acoustic waves in a solid medium with boundaries

  (e.g. plates, pipes, rods etc)
  - Energy of wave guided by the boundaries of the structure, hence terms <u>guided wave</u> and <u>waveguide</u>
  - <u>Dispersion curves:</u> (1 mm thick steel plate)



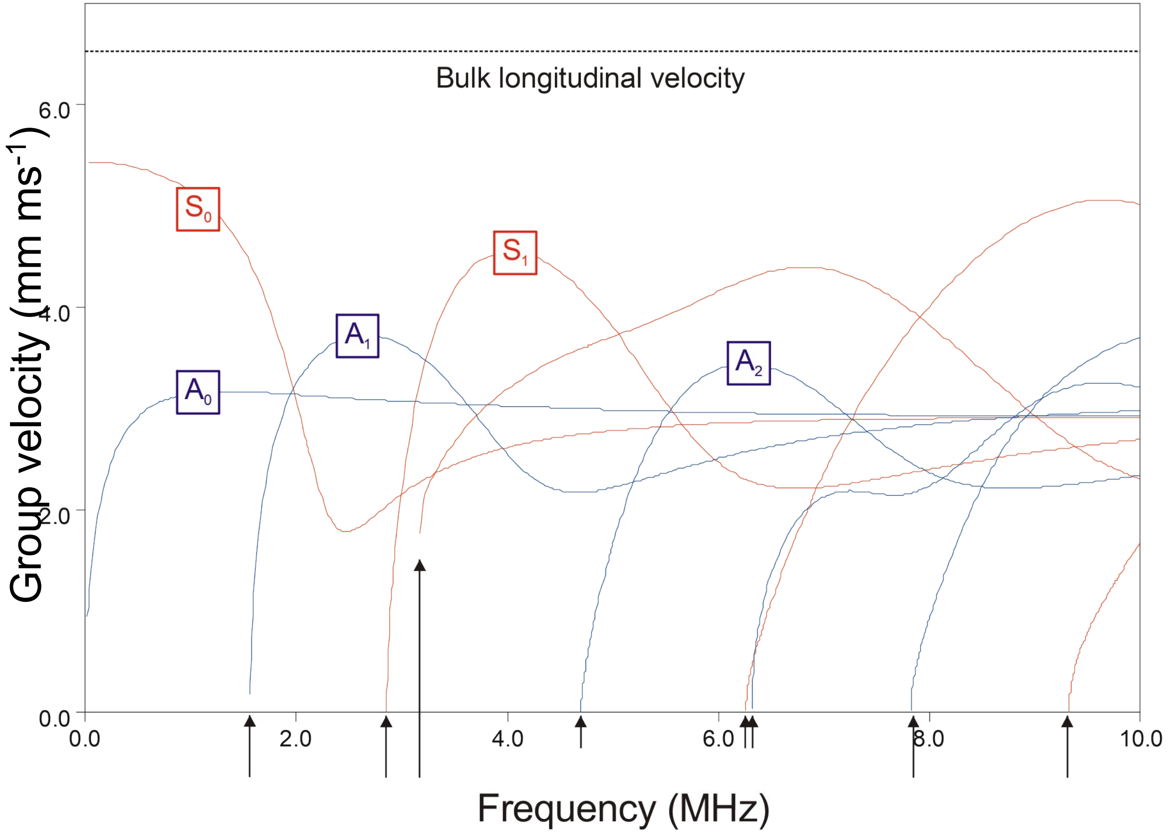
Key relationships and dispersion curves

- Example phase velocity dispersion curves



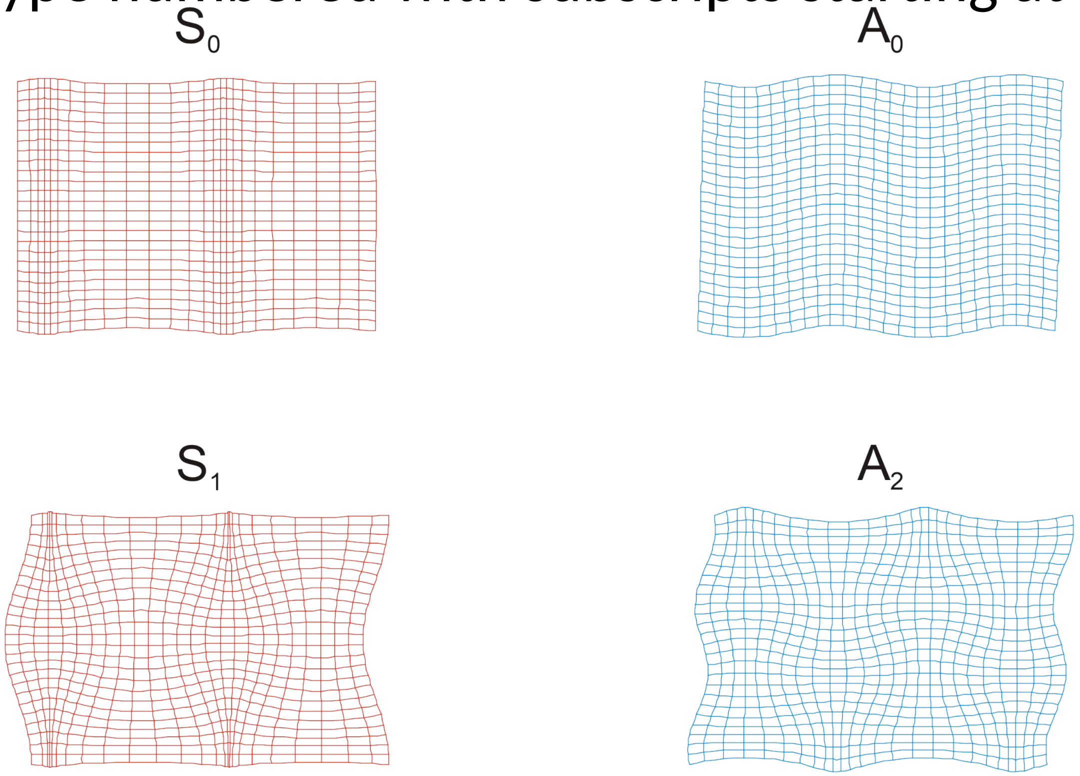
Key relationships and dispersion curves

- Example group velocity dispersion curves
  - Note no velocity exceeds bulk longitudinal wave velocity



Key relationships and dispersion curves

- Mode names
  - For flat plates modes are classified as symmetric (S) or antisymmetric (A) depending on their mode shape
  - Modes of each type numbered with subscripts starting at zero



Key characteristics relevant to us

- Complications
  - Multiple modes – signals can be hard to interpret
  - Multiple directions of propagation cf. bulk wave testing
  - Dispersion – pulses distort and lengthen as they propagate
  - Interaction with defects harder to quantify

How can we model these systems

Can’t carry out all the experiments we would like and need to demonstrate understanding of overall physics and behaviour of system

- Analytic models
- Numerical models

Analytic models

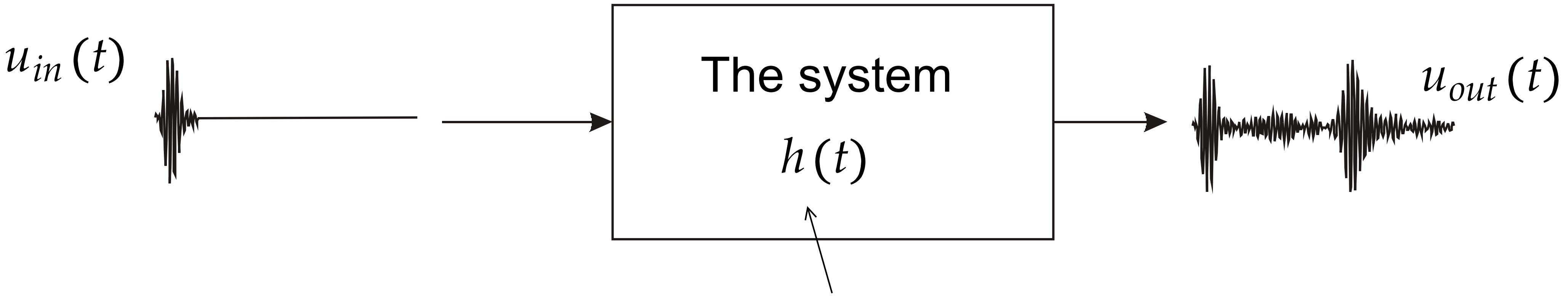
- In 2D and 3D there are exact solutions for certain specific cases, for example
  - Fields from point and line sources in infinite media
  - Plane waves obliquely incident on interfaces
  - Fields from simple transducer shapes in infinite media
  - Scattering of incident plane waves by simple defect shapes
  - 2D modal solution to propagation in an infinitely long flat plate (Lamb waves)
- But there is no general exact solution for arbitrary boundary conditions

Analytic models

- Modular approach
  - Break the system down into parts and model each part separately, e.g.
    - Beam profile from transducer
    - Interaction of plane waves with defect
    - Interaction of plane waves with boundaries
  - This is a good approach if interactions between different parts of system can be ignored (e.g. multiple scattering)

Linear Systems Modelling

- Propagation of ultrasound through any structure can be thought of as system with transfer function



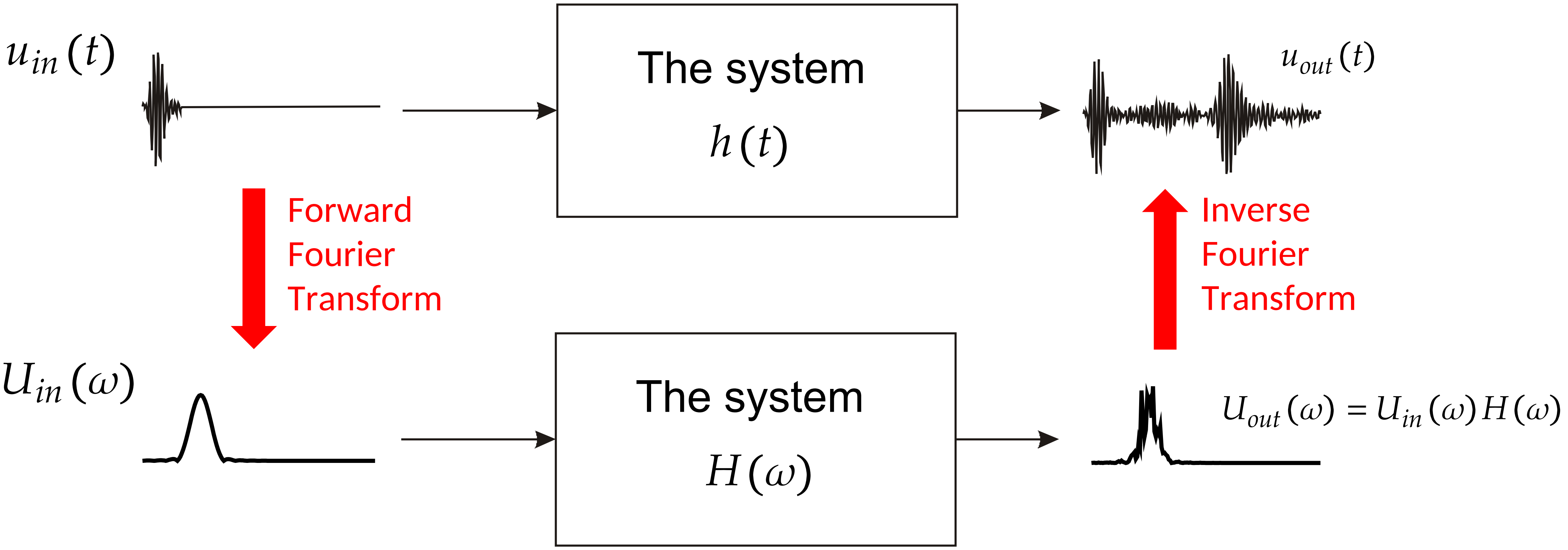
Transfer Function (impulse response in time-domain)

  - Transfer function describes everything that happens in the system
    - Transmitter characteristics, wave propagation, scattering, reflections, attenuation, receiver characteristics etc.
  - In time-domain, output signal for given input is obtained by convolution

$$u_{out}(t) = (u_{in} \otimes h)(t) = \int_{-\infty}^{\infty} u_{in}(\tau)\, h(t-\tau)\, d\tau$$

Linear Systems Modelling

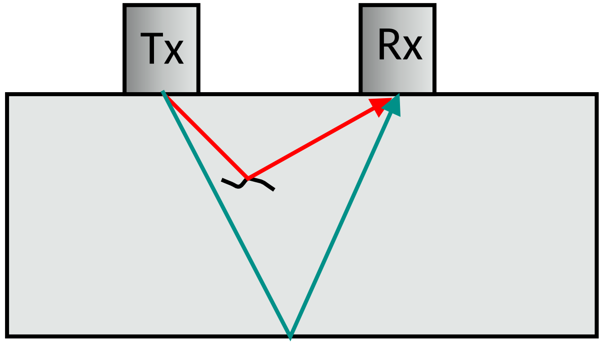
- In the frequency-domain, convolution is equivalent to multiplication



Typical Ultrasonic Transfer Functions, $H(\omega)$

- Transfer function is sum of transfer functions for each ray path from transmitter to receiver

$$H(\omega) = \sum_j H_j(\omega)$$



- For each ray path, transfer function typically looks like

$$H_j(\omega) = T_x(\omega)\, A(\omega)\, B X(\omega)\, \Delta(\omega)\, R_x(\omega)$$

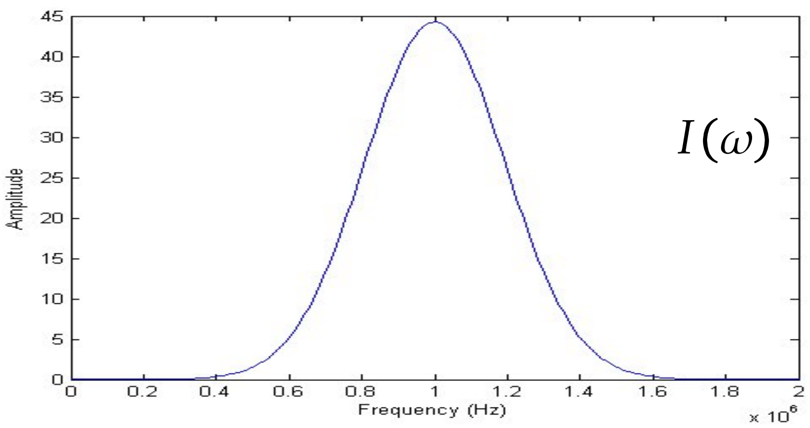
- where
  - $T_x(\omega)$ is the transmitting transducer characteristics
  - $A(\omega)$ is attenuation
  - B is beam spreading
  - $X(\omega)$ is the product of reflection and transmission coefficients encountered along ray path
  - $\Delta(\omega)$ is the time-delay due to propagation
  - $R_x(\omega)$ is the receiving transducer characteristics

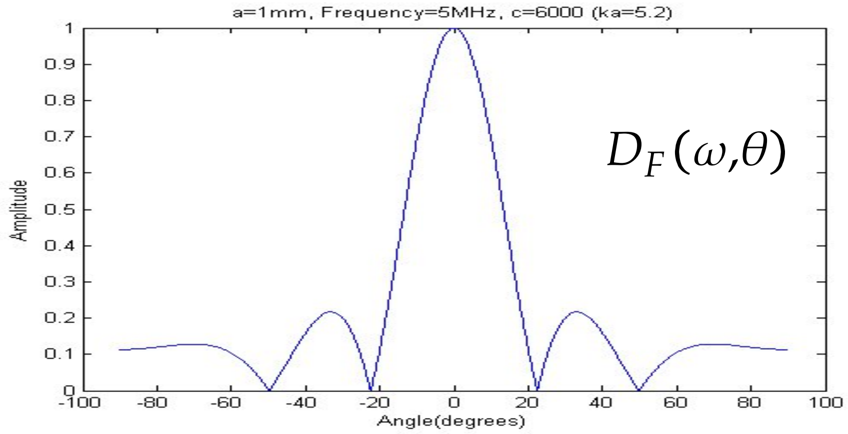
Typical Ultrasonic Transfer Functions, $H(\omega)$

- Transducer characteristics, $T_x(\omega)$ and $R_x(\omega)$
  - If same transducer is used for transmission and reception $T_x(\omega) = R_x(\omega)$
  - Behaviour of most transducers can be approximated as product of two effects:

$$T_x(\omega) = I(\omega)\, D_F(\omega,\theta)$$

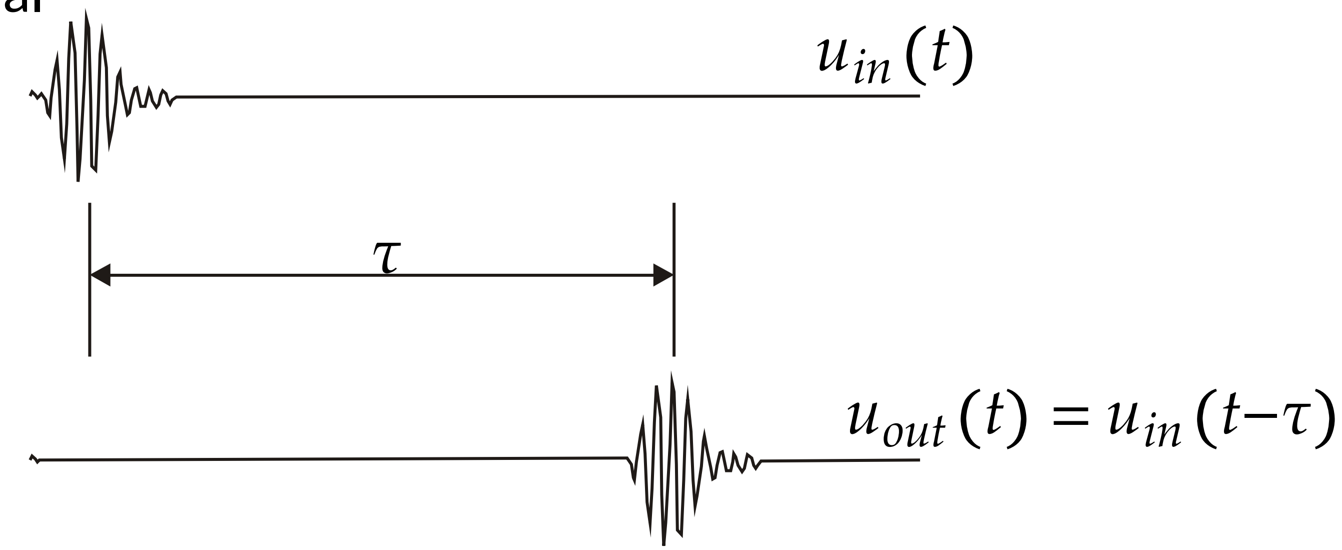
where $I(\omega)$ is the transducer frequency response characteristic, and $D_F(\omega,\theta)$ is the transducer directivity function which describes the transmitted amplitude (or reception sensitivity) to rays as a function of ray angle θ





Typical Ultrasonic Transfer Functions, $H(\omega)$

- Propagation term, $\Delta(\omega)$, for non-dispersive wave propagation
  - Non-dispersive wave propagation means that the received signal is just a delayed copy of the input signal



where $\tau = d/c$ where d is propagation distance and c is speed of sound

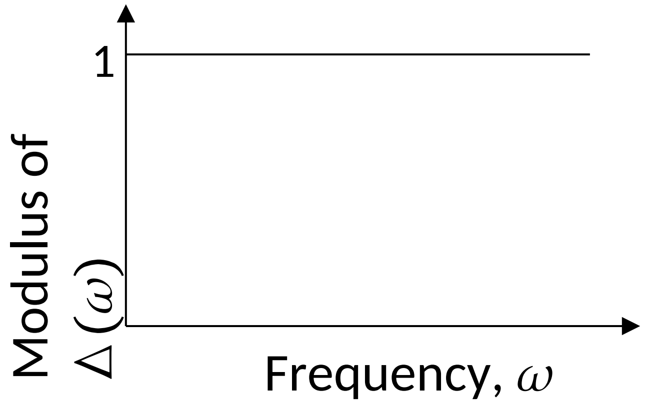
Typical Ultrasonic Transfer Functions, $H(\omega)$

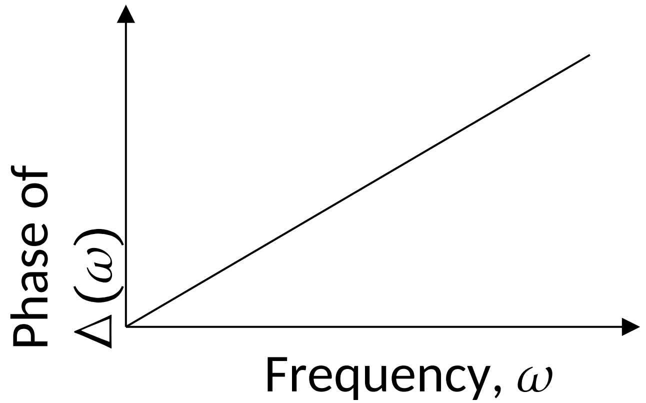
Fourier transform property of delayed signal:
if $\mathscr{F}\{g(t)\} = G(\omega)$ then $\mathscr{F}\{g(t-\tau)\} = G(\omega)\, e^{-i\omega\tau}$

- Here the time delay due to propagation over distance d is $\tau = d/c$, hence

$$\Delta(\omega) = e^{-i\omega\tau} = e^{-i\omega d/c} = e^{-ikd}$$

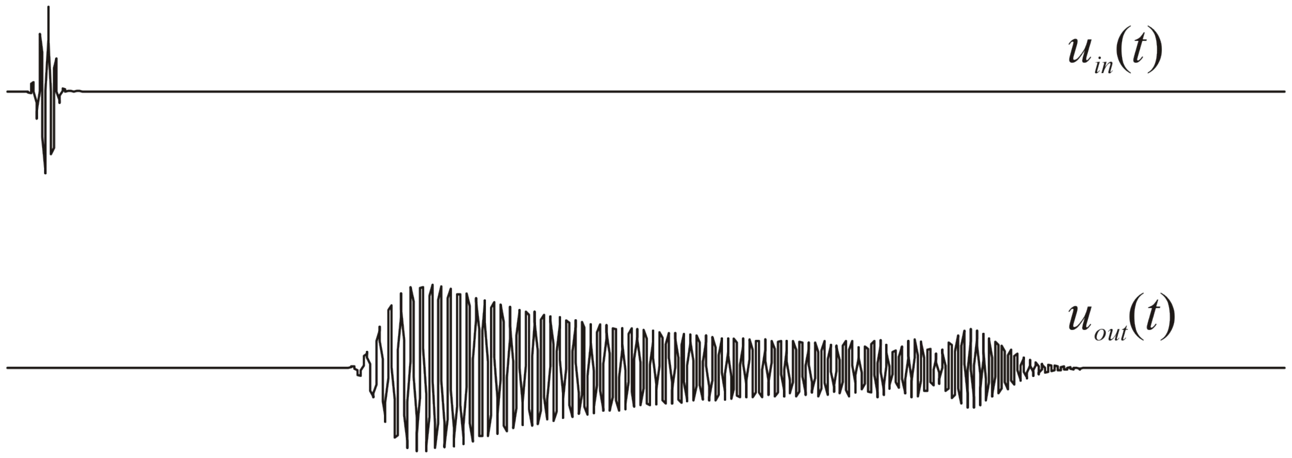
- Note $\Delta(\omega)$ does not modify amplitude spectrum; it only modifies the phase since $|\Delta(\omega)| = \left|e^{-i\omega\tau}\right| = 1$





Typical Ultrasonic Transfer Functions, $H(\omega)$

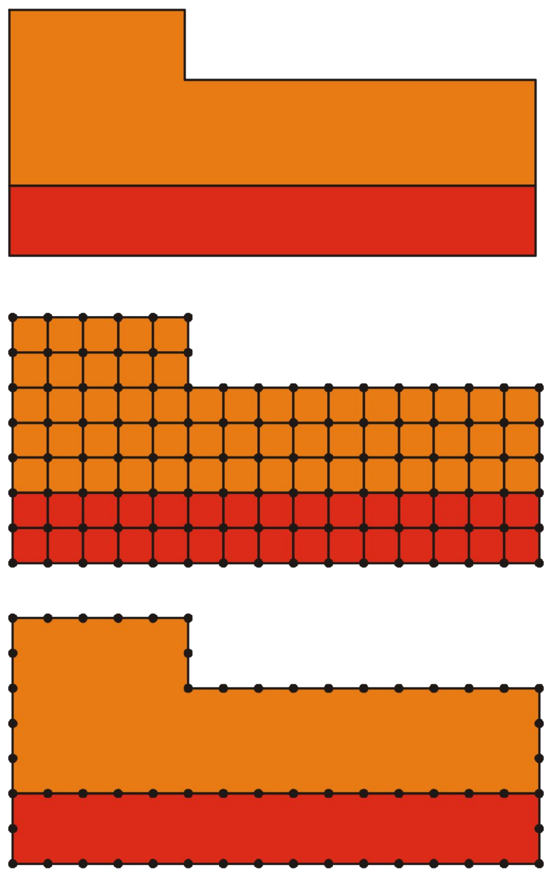
- Propagation term, $\Delta(\omega)$, for dispersive wave propagation
  - In some systems (e.g. guided waves) wave velocity is frequency dependent, i.e. $c = c(\omega)$
  - Hence signal distorts as it propagates as different frequencies travel at different velocities



  - This means $\Delta(\omega) = e^{-i\omega d/c(\omega)} = e^{-ik(\omega)d}$

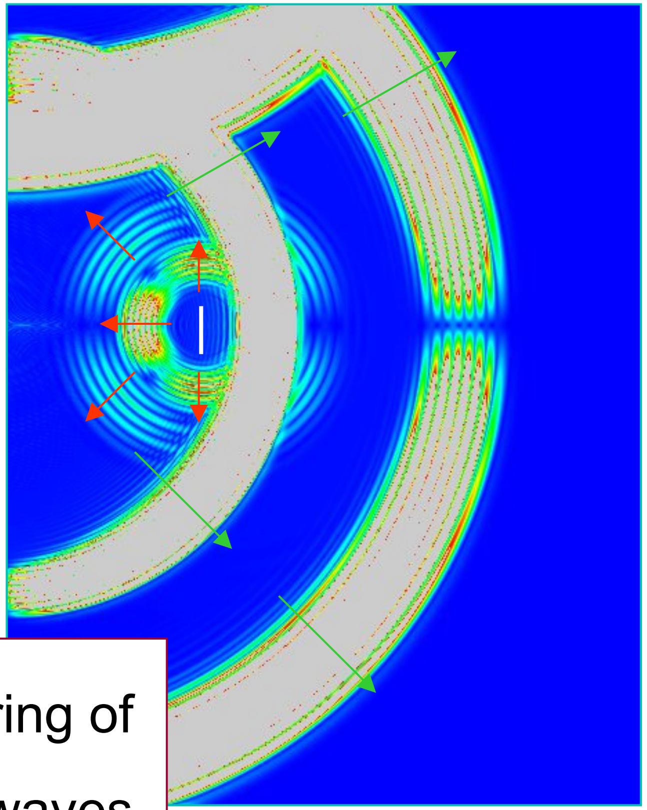
Numerical models

- The only general way to predict what happens in any system is by numerical modelling
- Discretisation of complete system into mesh
  - Finite difference method
  - Finite element (FE) method
  - Finite integration method
- Discretisation of system boundaries only into mesh
  - Boundary element method
    - More efficient than FE but more mathematical and less widely used; lack of commercial codes



Numerical models

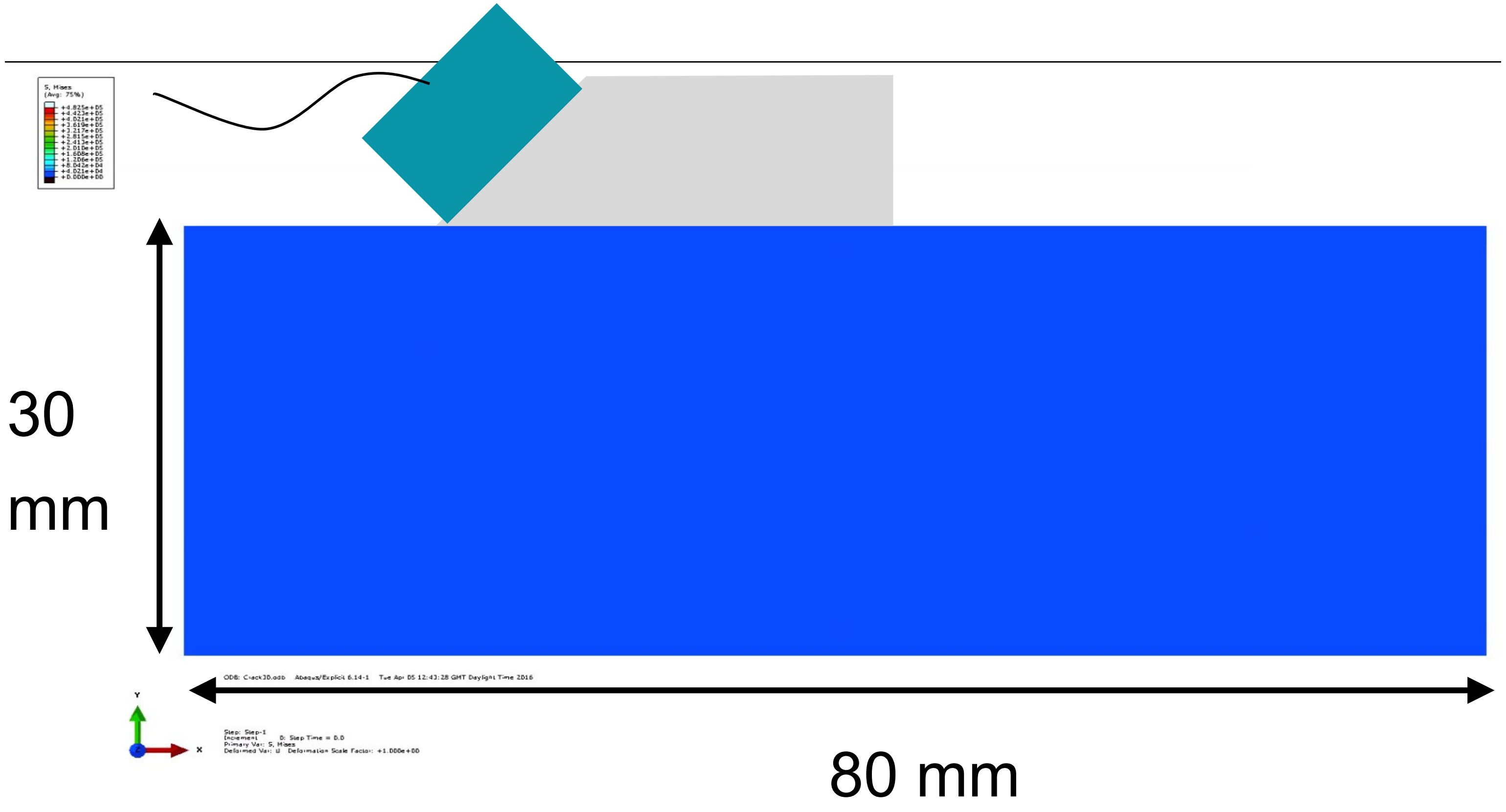
- Finite element modelling
  - Probably most widely used numerical method
  - Numerous commercial packages available (Abaqus, Ansys etc.)
  - Fairly intuitive
    - Usually performed in time-domain
    - Sound propagates through mesh like in real structure



Scattering of Lamb waves from a crack

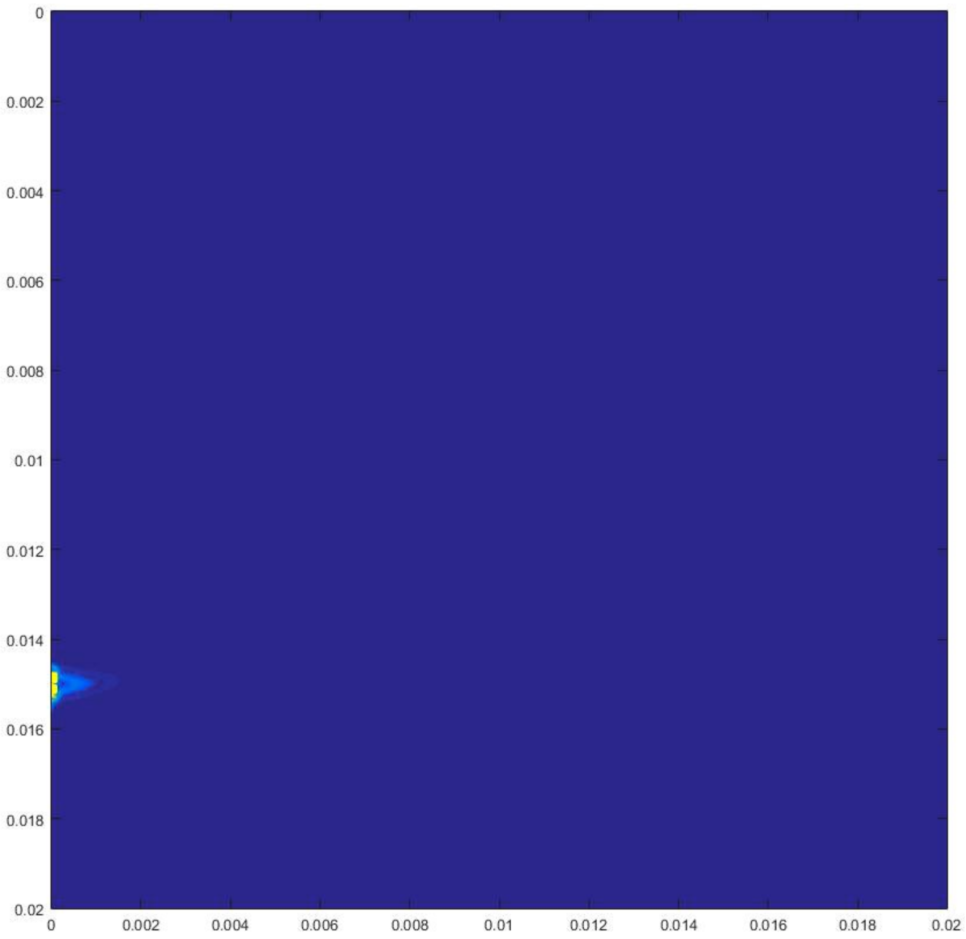
Examples of current modelling

- Full FE model
- 45degree bulk waves interacting with surface-breaking crack
  - 40 μs step time
  - ~ 5.5 million elements
  - ~ 60 s per model



Examples of current modelling

- Full FE model
- Bulk waves in curved composite showing anisotropic behaviour
    - 2 μs step time
    - ~ 1.2 million elements
    - ~ 3 s per model



Conclusions

- Fundamental theory and mathematics relatively straightforward
- Complexity grows very quickly
- Waves in bounded media can be very complex
- Modelling vital to support understanding
  - Analytical models, good and fast but ultimately only as good as assumptions
  - Numerical models increasingly used, often don’t help understanding